MATH2050C Assignment 6

Deadline: Feb 27

Hand in: 5, 12, 14

Section 3.5 no. 2, 3, 4, 5, 12, 13, 14.

Supplementary Exercises

1. Define the Fibonaci sequence by $f_{n+2} = f_{n+1} + f_n, f_1 = f_2 = 1$. Show that

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

   Hint: $(1 \pm \sqrt{5})/2$ are two roots of $x^2 = x + 1$ and f_n should be given by their linear combination.

See next page.

Fundamental Theorems

- Order Completeness Axiom: A nonempty set bounded from above has a supremum.
- Monotone Convergence Theorem: An increasing sequence converges provided it is bounded from above.
- Nested Interval Theorem: A family of nested closed, bounded intervals has a non-empty intersection.
- Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.
- Cauchy Completeness Theorem: Every Cauchy sequence converges.

Monotone Convergence Theorem was proved by Order Completeness Axiom and then it was used to prove Bolzano-Weierstrass Theorem. Another proof of Bolzano-Weierstrass Theorem is by the Nested Interval Theorem. On the other hand, Cauchy Completeness Theorem was proved by Bolzano-Weierstrass Theorem. In fact, all these five assertions are equivalent. More precisely, assuming any one of them, you can deduce the other four. Search Internet for more information if you are interested. In my college days, we were asked to drill on these proofs, but it has been out of fashion for many years. Finally, let me mention another assertion which plays the same role as the above five. It will become significant when we learn topology.

Heine-Borel Theorem: Let $I_n = (a_n, b_n), n \geq 1$, be open intervals covering $[a, b]$, that is, $[a, b] \subset \cup_n I_n$. Then one can pick finitely many I_n's to cover $[a, b]$.

Sterographic Projection

We have the following extended Bolzano-Weierstrass Theorem.

Theorem Every sequence has a subsequence which is either convergent or divergent to infinity.

A sequence is called divergent to infinity if for every $M > 0$, there is some n_0 such that $x_n \geq M$ for all $n \geq n_0$, or $x_n \leq -M$ for all $n \geq n_0$.

The proof is based on the sterographic projection. We first embed the real line to the plane as the x-axis. Consider the unit circle $\{(x, y) : \ x^2 + y^2 = 1\}$. A ray from the North Pole $(0, 1)$ downward would hit the unit circle at one point (x, y) and then the x-axis at $(a, 0)$ or it hits the x-axis at $(a, 0)$ first and then the unit circle at (x, y). The relation between (x, y) and a is given by

$$x = \frac{2a}{a^2 + 1}, \quad y = \frac{a^2 - 1}{a^2 + 1} \ ,$$

and

$$a = \frac{x}{1 - y} \ .$$

Thus points $(0, -1), (1, 0), (-1, 0)$ are mapped to $0, 1, -1$ correspondingly. This correspondence sets up a one-to-one onto map from the unit circle minus the North Pole to $\mathbb{R}$ where the North Pole is "mapped" to the positive and negative infinity in $\mathbb{R}$.

Now, we can prove the extended Bolzano-Weierstrass Theorem. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Put it on the x-axis as $(a_n, 0)$. Let its correspondence on the unit circle be (x_n, y_n). As the unit circle is a bounded set, by the two dimensional Bolzano-Weierstrass theorem there is a convergent subsequence (x_{n_k}, y_{n_k}) converging to (x, y). When $(x, y) \neq (0, 1)$, $y_{n_k} < 1$ for all large n_k, hence

$$a_{n_k} = \frac{x_{n_k}}{1 - y_{n_k}} \to \frac{x}{1 - y} \ .$$

In case $(x, y) \to (0, 1)$, WLOG assume that $x_{n_k} > 0$. We have

$$a_{n_k} = \frac{x_{n_k}}{1 - y_{n_k}} = \frac{\sqrt{1 - y_{n_k}^2}}{1 - y_{n_k}} = \sqrt{\frac{1 + y_{n_k}}{1 - y_{n_k}}} \to \infty \ ,$$

as $n_k \to \infty$. That is, $\{a_{n_k}\}$ diverges to infinity.